On the Three-Dimensional Orthogonal Drawing of Series-Parallel Graphs (Extended Abstract)

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Abstract—It has been known that every 6-graph has a 3-bend 3-D orthogonal drawing, while it has been open whether every 6-graph has a 2-bend 3-D orthogonal drawing. For the interesting open question, it is known that every 5-graph has a 2-bend 3-D orthogonal drawing, and every outerplanar 6-graph without triangles has a 0-bend 3-D orthogonal drawing. We show in this paper that every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing.

I. INTRODUCTION

We consider the problem of generating orthogonal drawings of graphs in the space. The problem has obvious applications in the design of 3-D VLSI circuits and optoelectronic integrated systems [7], [11]. Throughout this paper, we consider simple connected graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex $v$ in $G$, and by $\Delta(G)$ the maximum degree of a vertex of $G$. $G$ is called a $k$-graph if $\Delta(G) \leq k$. It is well-known that every graph can be drawn in the space so that its edges intersect only at their ends. Such a drawing of a graph $G$ is called a 3-D drawing of $G$. A 3-D orthogonal drawing of $G$ is a 3-D drawing such that each edge is drawn by a sequence of contiguous axis-parallel line segments. Notice that a graph $G$ has a 3-D orthogonal drawing only if $\Delta(G) \leq 6$. A 3-D orthogonal drawing with no more than $b$ bends per edge is called a $b$-bend 3-D orthogonal drawing.

Eades, Symvonis, and Whitesides [4], and Papakostas and Tollis [10] showed that every 6-graph has a 3-bend 3-D orthogonal drawing. Eades, Symvonis, and Whitesides [4] also posed an interesting open question of whether every 6-graph has a 2-bend 3-D orthogonal drawing. Wood [14] showed that every 5-graph has a 2-bend 3-D orthogonal drawing. Nomura, Tayu, and Ueno [9] showed that every outerplanar 6-graph has a 0-bend 3-D orthogonal drawing if and only if it contains no triangle as a subgraph, while Eades, Stirk, and Whitesides [3] proved that it is $\mathcal{NP}$-complete to decide if a given 5-graph has a 0-bend 3-D orthogonal drawing. We show in this paper the following theorem.

Theorem 1: Every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing.

The proof of Theorem 1 is constructive and provides a polynomial time algorithm to generate such a drawing for a series-parallel 6-graph.

It is still open whether every 6-graph has a 2-bend 3-D orthogonal drawing. It is also open whether every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing.

II. SERIES-PARALLEL GRAPHS

A series-parallel graph is defined recursively as follows.

1. A graph consisting of two vertices joined by a single edge is a series-parallel graph. The vertices are the terminals.
2. If $G_1$ is a series-parallel graph with terminals $s_1$ and $t_1$, and $G_2$ is a series-parallel graph with terminals $s_2$ and $t_2$, then a graph $G$ obtained by either of the following operations is also a series-parallel graph:
   (i) Series-composition: identify $t_1$ with $s_2$. Vertices $s_1$ and $t_2$ are the terminals of $G$.
   (ii) Parallel-composition: identify $s_1$ and $s_2$ into a vertex $s$, and $t_1$ and $t_2$ into $t$. Vertices $s$ and $t$ are the terminals of $G$.

III. 3-D EMBEDDINGS AND ORTHOGONAL DRAWINGS

The three-dimensional (3-D) grid $G$ is an (infinite) graph consisting of $\mathbb{Z}^3$, the set of grid-points in 3-D space with integer coordinates, together with the axis-parallel edges connecting neighboring grid-points. The grid-points are also considered as vectors. The 3-D embedding $\langle \phi, \rho \rangle$ of a graph $G$ is defined by a one-to-one mapping $\phi : V(G) \rightarrow V(\mathbb{Z}^3)$, together with a mapping $\rho$ that maps each edge $(u, v) \in E(G)$ onto a path $\rho(u, v)$ in $G$ that connects $\phi(u)$ and $\phi(v)$. A path $P$ in $G$ is called a $k$-bend path if $P$ contains $k$ bends.

Let $D^+ = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $D^- = \{(-1, 0, 0), (0, -1, 0), (0, 0, -1)\}$, and $D = D^+ \cup D^-$. A vector in $D$ is called a direction.

Let $(\phi, \rho)$ be a 3-D embedding of a graph $G$, and $(u, v) \in E(G)$. If $g$ is a grid-point adjacent with $\phi(u)$ in path $\rho(u, v)$ in $G$, there exists a direction $d \in D$ such that $g = \phi(u) + d$. For the two-dimensional case, Biedl and Kant [2], and Liu, Morgana, and Simeone [8] showed that every planar 4-graph has a 2-bend 2-D orthogonal drawing with only exception of the octahedron. Moreover, Kant [6] showed that every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception of $K_4$. Tayu, Nomura, and Ueno [12] showed that every series-parallel 4-graph has a 1-bend 2-D orthogonal drawing. Nomura, Tayu, and Ueno [9] showed that every outerplanar 3-graph has a 0-bend 2-D orthogonal drawing if and only if it contains no triangle as a subgraph. On the other hand, Garg and Tamassia [5] proved that it is $\mathcal{NP}$-complete to decide if a given planar 4-graph has a 0-bend 2-D orthogonal drawing. Battista, Liotta, and Vargiu [1] showed that the problem can be solved in polynomial time for planar 3-graphs and series-parallel graphs.

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We denote such $d$ by $\alpha^u_d(e)$. It is easy to see the following lemma.

Lemma 1: If $\rho(u, v)$ is a $2$-bend path, $\rho(u, v)$ is uniquely determined by $\phi(u), \phi(v), \alpha^u_d(e),$ and $\alpha^v_d(e).$

Figure 1 shows a $2$-bend path $\rho(u, v)$ determined by $\phi(u) = (0, 0, 0), \phi(v) = (3, 2, 1), \alpha^u_d(e) = (0, 0, 1),$ and $\alpha^v_d(e) = (-1, 0, 0).

![Fig. 1. Example of a 2-bend path $\rho(u, v)$.](Image)

Two grid-points $g$ and $g + (a, b, c)$ are said to be in the general position if $abc \neq 0$. Let $g_0, g_1 = g_0 + (x_1, y_1, z_1),$ and $g_2 = g_0 + (x_2, y_2, z_2)$ be grid-points in the general position. Then, we define $g_i \leq g_0$ if $|w_1| < |w_2|$ or $|w_1|w_2 < 0$ for any $w \in \{x, y, z\}$. A $3$-D embedding is called a $\tau$-embedding if all of the following conditions are satisfied:

**Condition A:** If $u \neq v$ then $\phi(u)$ and $\phi(v)$ are in the general position.

**Condition B:** For any distinct edges $e, e' \in E(G)$ incident to a vertex $u$, $\alpha^u_d(e) \neq \alpha^u_d(e')$.

**Condition C:** $\rho(e)$ is a $2$-bend path for any $e \in E(G)$.

**Condition D:** For any edges $e_1 = (u, v_1)$ and $e_2 = (u, v_2)$ incident to a vertex $u$, $(D-1)$ or $(D-2)$ below holds;

- $(D-1)$ $\alpha^u_d(e_1) = \pm \alpha^u_d(e_2)$;
- $(D-2)$ $\phi(v_1) \leq_{\phi(u)} \phi(v_2)$ or $\phi(v_2) \leq_{\phi(u)} \phi(v_1)$.

It follows from Condition B that $G$ has a $\tau$-embedding only if $G$ is a 6-graph. The purpose of this section is to show the following theorem.

**Theorem 2:** A $\tau$-embedding of a 6-graph $G$ induces a $2$-bend 3-D orthogonal drawing of $G$.

**Proof (Sketch):** The theorem is proved by Lemma 1 together with the following two lemmas.

**Lemma 2:** Let $(\phi, \rho)$ be a 3-D embedding of $G$ satisfying Conditions A through C. If edges $e_1$ and $e_2$ have no common paths, $\rho(e_1)$ and $\rho(e_2)$ are disjoint.

**Proof:** Omitted in the extended abstract.

**Lemma 3:** Let $(\phi, \rho)$ be a 3-D embedding of $G$ satisfying Conditions A through C. If any adjacent edges $e_1 = (u, v_1)$ and $e_2 = (u, v_2)$ satisfy Condition D, $\rho(e_1)$ and $\rho(e_2)$ are internally disjoint.

**Proof:** Omitted in the extended abstract.

**IV. PROOF OF THEOREM 1 (SKETCH)**

Let $G$ be a series-parallel 6-graph with terminals $s$ and $t$. Before proving the theorem, we need some preliminaries.

**IV-A. 9-CUBIC 3-D EMBEDDINGS**

Let $p = (p_x, p_y, p_z)$ and $q = (q_x, q_y, q_z)$ be grid-points in the general position, and let $g_{\min}(p, q) = \min\{p_w, q_w\}$ and $g_{\max}(p, q) = \max\{p_w, q_w\}$ for each $w \in \{x, y, z\}$. A 3-D sub-grid $Q_{p, q}$ induced by a set of grid points $\{(i_x, i_y, i_z)\}_{i_x, i_y, i_z} \leq i_x \leq g_{\max}(p, q), g_{\min}(p, q) \leq i_y \leq g_{\max}(p, q), g_{\min}(p, q) \leq i_z \leq g_{\max}(p, q)\}$ is called a center-cube for $p$ and $q$. Let $q = p + (a, b, c)$. For each $\sigma \subseteq \{x, y, z\}$, define grid points $g_{\sigma}$ as follows:

- For each $\sigma$, $g_{\sigma}$ corresponds to a corner of $Q_{p, q}$.

**Fig. 2. Cubes for $p$ and $q$.**

![Fig. 2. Cubes for $p$ and $q$.](Image)

For each $\sigma \subseteq \{x, y, z\}$, $Q_{p, q}^\sigma$ is a 3-D subgrid induced by a vertex set $\{(i_x, i_y, i_z) | i_x \in X_{\sigma}, i_y \in Y_{\sigma}, i_z \in Z_{\sigma}\}$. A 3-D grid $Q_{p, q}^\sigma$ is called a corner-cube for $p$ and $q$. We define that $V_{p, q} = V(Q_{p, q}) \cup \bigcup_{\sigma \subseteq \{x, y, z\}} V(Q_{p, q}^\sigma)$. Figure 2 illustrates an example of $Q_{p, q}$ and corner cubes $Q_{p, q}^\sigma$.
IV-B. FEASIBLE PAIR

For two vectors \( a = (a_x, a_y, a_z) \) and \( b = (b_x, b_y, b_z) \), we define that \( a \cdot b = (a_x b_x + a_y b_y + a_z b_z) \). We denote by \( a \cdot b \) the inner product of \( a \) and \( b \). A vector \( r \in \{-1, 1\}^3 \) is called a diagonal direction. For a diagonal direction \( r \), let \( D_r^+ = \{(1,0,0) + r, (0,1,0) + r, (0,0,1) + r\} \) and \( D_r^- = \mathcal{D} - D_r^+ \). It should be noted that \( d \cdot r \in D_r^+ \) if and only if \( d \in D_r^+ \) and \( d \cdot r = 1 \). Also, \( d \cdot r \in D_r^- \) if and only if \( d \in D_r^- \) and \( d \cdot r = -1 \).

For any \( D_1, D_2 \subseteq \mathcal{D} \), if \( D_1 \) is said to be non-admissible if \( D_1 = \{ d \} \) and \( D_2 = \{-d\} \) for some \( d \in \mathcal{D} \). Otherwise, \( \langle D_1, D_2 \rangle \) is said to be admissible.

For \( D_1, D_2 \subseteq \mathcal{D} \) and a diagonal direction \( r \), \( \langle D_1, D_2 \rangle \) is said to be inner-directed for \( r \) if there exist directions \( d_s \in D_s \cap D_r^+ \) and \( d_t \in D_t \cap D_r^- \) such that \( d_s \cdot d_t = 0 \), and \( \{ D_s - \{ d_s \}, D_t - \{ d_t \} \} \) is admissible.

For a series-parallel 6-graph with terminals \( s \) and \( t \), a diagonal direction \( r \), and \( D_s, D_t \subseteq \mathcal{D} \), \( \langle D_s, D_t \rangle \) is said to be feasible for \( G \) and \( r \) if all of the following conditions are satisfied:

1. \( |D_s| = d_G(s) \).
2. \( |D_t| = d_G(t) \).
3. \( (D_s, D_t) \) is inner-directed for \( r \) if \( (s, t) \in E(G) \), and \( (D_s, D_t) \) is admissible if \( (s, t) \notin E(G) \).

It should be noted that if \( \langle D_s, D_t \rangle \) is feasible for some \( G \) and \( r \in \{-1, 1\}^3 \), then \( \langle D_s, D_t \rangle \) is also admissible.

It is easy to see the following.

**Lemma 4:** For any series-parallel 6-graph \( G \) and any diagonal direction \( r \in \{-1, 1\}^3 \), there exist \( D_s, D_t \subseteq \mathcal{D} \) such that \( \langle D_s, D_t \rangle \) is feasible for \( G \) and \( r \).

IV-C. PROOF

For any grid-points \( p \) and \( q = p + (a, b, c) \) in the general position, a diagonal direction \( \langle a/|a|, b/|b|, c/|c| \rangle \) is denoted by \( R_{p,q} \). Now, we are ready to prove the following.

**Theorem 3:** For a series-parallel 6-graph \( G \) with terminals \( s \) and \( t \), a diagonal direction \( r \), and \( D_s, D_t \subseteq \mathcal{D} \) such that \( \langle D_s, D_t \rangle \) is feasible for \( G \) and \( r \), there exists a 9-cubic \( \tau \)-embedding \( \langle \phi, \rho \rangle \) of \( G \) such that \( \{ \alpha_\phi(e) \}_{e \in E(G)} = D_s \), \( \{ \alpha_\rho(e) \}_{e \in E(G)} = D_t \), and \( \mathcal{R}_{\phi(s), \phi(t)} = r \).

The proof of Theorem 3 is shown in the next section. Theorem 1 follows from Theorems 2 and 3, and Lemma 4.

V. PROOF OF THEOREM 3 (SKETCH)

The theorem is proved by induction on \( |E(G)| \).

If \( |E(G)| = 1 \), \( G \) is a graph consisting of only one edge \((s,t)\) and so \( |D_s| = |D_t| = 1 \). Since \((s,t) \in E(G)\) and \( \langle D_s, D_t \rangle \) is feasible for \( G \) and a diagonal direction \( r \), \( \langle D_s, D_t \rangle \) is said to be inner-directed for \( r \). Without loss of generality we assume that \( r = (1,1,1) \), \( d_s = (1,0,0) \), and \( d_t = (0,-1,0) \). Define a 3-D embedding \( \langle \phi, \rho \rangle \) of \( G \) as follows: \( \phi(s) = (0,0,0) \), \( \phi(t) = (1,1,1) \), and \( \rho(s,t) \) is a path connecting \( \phi(s) \) and \( \phi(t) \), and passing through \((1,0,0)\) and \((1,0,1)\) as shown in Fig. 3. It is easy to see that \( \langle \phi, \rho \rangle \) is a 9-cubic \( \tau \)-embedding of \( G \).

V.A. CASE 1: PARALLEL-COMPOSITION

We consider the case when \( G \) is a parallel-composition of series-parallel graphs \( G_1 \) and \( G_2 \). We denote the terminals of \( G_1 \) and \( G_2 \) by \( s \) and \( t \). We further distinguish two cases.

**Case 1-1** \((s,t) \in E(G)\).

Without loss of generality, we assume that \( G_1 \) consists of exactly one edge \( (s,t) \) and \( G_2 \) is the graph obtained from \( G \) by deleting the edge \( (s,t) \). Then, \( \langle D_s, D_t \rangle \) is inner-directed for \( r \) since \( \langle D_s, D_t \rangle \) is feasible for \( G \) and \( r \).

**Lemma 5:** There exist \( d_s \in D_s \) and \( d_t \in D_t \) such that \( \{ \langle d_s \rangle, \langle d_t \rangle \} \) is feasible for \( G_1 \) and \( r \), and \( \langle D_s - \{ d_s \}, D_t - \{ d_t \} \rangle \) is feasible for \( G_2 \) and \( r \).

Thus, by the induction hypothesis, \( G_1 \) has a 9-cubic \( \tau \)-embedding \( \langle \phi_1, \rho_1 \rangle \) for \( r \). Also, \( G_2 \) has a 9-cubic \( \tau \)-embedding \( \langle \phi_2, \rho_2 \rangle \) for \( r \).

We can prove that we can construct a 9-cubic \( \tau \)-embedding \( \langle \phi, \rho \rangle \) of \( G \) for \( \langle D_s, D_t \rangle \) and \( r \) from \( \langle \phi_1, \rho_1 \rangle \) and \( \langle \phi_2, \rho_2 \rangle \).

**Case 1-2** \((s,t) \notin E(G)\).

We can prove the following.

**Lemma 6:** \( D_v \) can be partitioned into \( D_v^{(1)} \) and \( D_v^{(2)} \) for \( v \in \{ s, t \} \) such that \( \langle D_v^{(1)}, D_v^{(1)} \rangle \) is feasible for \( G_1 \) and \( r \), and \( \langle D_v^{(2)}, D_v^{(2)} \rangle \) is feasible for \( G_2 \) and \( r \).

Thus, by the induction hypothesis, \( G_i (i = 1, 2) \) has a 9-cubic \( \tau \)-embedding \( \langle \phi_i, \rho_i \rangle \) for \( (D_v^{(1)}, D_v^{(2)}) \) and \( r \).

We can prove that we can construct a 9-cubic \( \tau \)-embedding \( \langle \phi, \rho \rangle \) of \( G \) for \( \langle D_s, D_t \rangle \) and \( r \) from \( \langle \phi_1, \rho_1 \rangle \) and \( \langle \phi_2, \rho_2 \rangle \).

V.B. CASE 2: SERIES-COMPOSITION

We consider the case when \( G \) is a series-composition of series-parallel graphs \( G_1 \) and \( G_2 \). Without loss of generality, we denote the terminals of \( G_1 \) by \( s \) and \( u \), and those of \( G_2 \) by \( u \) and \( t \).

Since \( \langle D_s, D_t \rangle \) is admissible, there exist \( d_s \in D_s \) and \( d_t \in D_t \) satisfying \( d_s \neq -d_t \). We further distinguish three cases.

**Case 2-1** \( d_s \in D_r^+ \) and \( d_t \in D_r^- \).

We can prove the following.

**Lemma 7:** There exist disjoint sets \( D_u^{(s)} \) and \( D_u^{(t)} \) of directions such that \( \langle D_s, D_u^{(s)} \rangle \) is feasible for \( G_1 \) and \( r \), and \( \langle D_t, D_u^{(t)} \rangle \) is feasible for \( G_2 \) and \( r \).
Thus, by the induction hypothesis, $G_1$ has a 9-cubic $\tau$-embedding $\langle \phi^{(1)}, \rho^{(1)} \rangle$ for $\langle D_s, D_u^{(s)} \rangle$ and $r$, and $G_2$ has a 9-cubic $\tau$-embedding $\langle \phi^{(2)}, \rho^{(2)} \rangle$ for $\langle D_s^{(2)}, D_2 \rangle$ and $r$.

We can prove that we can construct a 9-cubic $\tau$-embedding $\langle \phi, \rho \rangle$ of $G$ for $\langle D_s, D_t \rangle$ and $r$ from $\langle \phi^{(1)}, \rho^{(1)} \rangle$ and $\langle \phi^{(2)}, \rho^{(2)} \rangle$.

**Case 2-2** $d_s \in D^+_r$ and $d_t \in D^+_r$ or $d_s \in D^-_r$ and $d_t \in D^-_r$.

It should be noted that $d_s \cdot r = d_t \cdot r$. Let $r_s = r$ if $d_s \cdot r = 1$ and $r_s = -r$ otherwise, and let $r_u = -r_s$. We can prove the following.

**Lemma 8:** There exist disjoint sets $D_s^{(s)}$ and $D_u^{(s)}$ of directions such that $\langle D_s, D_s^{(s)} \rangle$ is feasible for $G_1$ and $r_s$ and $\langle D_u^{(s)}, D_t \rangle$ is feasible for $G_2$ and $r_u$.

Thus, by the induction hypothesis, $G_1$ has a 9-cubic $\tau$-embedding $\langle \phi^{(1)}, \rho^{(1)} \rangle$ for $\langle D_s, D_s^{(s)} \rangle$ and $r_s$, and $G_2$ has a 9-cubic $\tau$-embedding $\langle \phi^{(2)}, \rho^{(2)} \rangle$ for $\langle D_t, D_t \rangle$ and $r_u$.

We can prove that we can construct a 9-cubic $\tau$-embedding $\langle \phi, \rho \rangle$ of $G$ for $\langle D_s, D_t \rangle$ and $r$ from $\langle \phi^{(1)}, \rho^{(1)} \rangle$ and $\langle \phi^{(2)}, \rho^{(2)} \rangle$.

**Case 2-3** $d_s \in D^-_r$ and $d_t \in D^+_r$.

Let $r_s = r + 2d_s$ and $r_u = -r_s$. It should be noted that $r_s \in \{-1,1\}^2$, since $d_s \cdot r = -1$. We can prove the following.

**Lemma 9:** There exist disjoint sets $D_s^{(s)}$ and $D_u^{(s)}$ of directions such that $\langle D_s, D_s^{(s)} \rangle$ is feasible for $G_1$ and $r_s$ and $\langle D_u^{(s)}, D_t \rangle$ is feasible for $G_2$ and $r_u$.

Thus, by the induction hypothesis, $G_1$ has a 9-cubic $\tau$-embedding $\langle \phi^{(1)}, \rho^{(1)} \rangle$ for $\langle D_s, D_s^{(s)} \rangle$ and $r_s$, and $G_2$ has a 9-cubic $\tau$-embedding $\langle \phi^{(2)}, \rho^{(2)} \rangle$ for $\langle D_t^{(2)}, D_t \rangle$ and $r_u$.

We can prove that we can construct a 9-cubic $\tau$-embedding $\langle \phi, \rho \rangle$ of $G$ for $\langle D_s, D_t \rangle$ and $r$ from $\langle \phi^{(1)}, \rho^{(1)} \rangle$ and $\langle \phi^{(2)}, \rho^{(2)} \rangle$.

This completes the proof of Theorem 3.

**VI. EXAMPLES**

A series-parallel 6-graph $G$ shown in Fig. 4(a) is a parallel-composition of series-parallel 6-graphs $G_1$ and $G_2$ shown in Fig. 4(b) and (c), respectively. Given $D_s = \{\{-1,0,0\},\{0,0,-1\}\}, D_t = \{\{-1,0,0\},\{0,-1,0\}\}$, and a diagonal direction $r = (-1,1,1)$, there exist $d_s = (-1,0,0)$ and $d_t = (0,-1,0)$ such that $\langle \{d_s\}, \{d_t\}\rangle$ is feasible for $G_1$ and $r$, and $\langle \{0,0,-1\},\{-1,0,0\}\rangle$ is feasible for $G_2$ and $r$ by Lemma 5.

A 9-cubic $\tau$-embedding $\langle \phi^{(1)}, \rho^{(1)} \rangle$ of $G_1$ for $\langle \{d_s\}, \{d_t\}\rangle$ and $r$ is shown in Fig. 4(e), and a 9-cubic $\tau$-embedding $\langle \phi^{(2)}, \rho^{(2)} \rangle$ of $G_2$ for $\langle \{0,0,-1\},\{-1,0,0\}\rangle$ and $r$ is shown in Fig. 4(f). We obtain a 9-cubic $\tau$-embedding $\langle \phi, \rho \rangle$ of $G$ for $\langle D_s, D_t \rangle$ and $r$ from $\langle \phi^{(1)}, \rho^{(1)} \rangle$ and $\langle \phi^{(2)}, \rho^{(2)} \rangle$ as shown in Fig. 4(g). The 9-cubic $\tau$-embedding $\langle \phi, \rho \rangle$ of $G$ induces a 2-bend 3-D orthogonal drawing of $G$ by Theorem 2.

**REFERENCES**


